(b) The first term is 20 and the quotient is $1/1.2$, so the sum is $\dfrac{20}{1-1/1.2} = 120$. (c) $\dfrac{3}{1-2/5} = 5$

(d) The first term is $(1/20)^{-2} = 400$ and the quotient is $1/20$, so the sum is $\dfrac{400}{1-1/20} = 8000/19$.

(a) $\int_0^T ae^{-rt}\,dt = (a/r)(1-e^{-rT})$ (b) a/r, the same as (11.5.4).

$5000(1.04)^4 = 5849.29$

9. 21 232.32

10. $K \approx 5990.49$

11. (a) According to formula (11.6.2), the annual payment is: $500\,000 \cdot 0.07(1.07)^{10}/(1.07^{10}-1) \approx 71\,188.80$. The total amount is $10 \cdot 71\,188.80 = 711\,888$. (b) If the person has to pay twice a year, the biannual payment is $500\,000 \cdot 0.035(1.035)^{20}/(1.035^{20}-1) \approx 35\,180.50$. The total amount is then $20 \cdot 35\,180.50 = 703\,610.80$.

12. (a) The present value is $(3200/0.08)[1-(1.08)^{-10}] = 21\,472.26$.

(b) The present value is $7000 + (3000/0.08)[1-1.08^{-5}] = 18\,978.13$.

(c) Four years ahead the present value is $(4000/0.08)[1-(1.08)^{-10}] = 26\,840.33$. The present value when Lucy makes her choice is $26\,840.33 \cdot 1.08^{-4} = 19\,728.44$. So she should choose option (a).

13. (a) $t^* = 1/16r^2 = 25$ for $r = 0.05$. (b) $t^* = 1/\sqrt{r} = 5$ for $r = 0.04$.

14. (a) The total revenue is $F(10){=}F(10) - F(0) = \int_0^{10} (1+0.4t)\,dt = \Big|_0^{10} (t+0.2t^2) = 30$. (b) See Example 10.5.3.

15. (a) $x_t = (-0.1)^t$ (b) $x_t = -2t+4$ (c) $x_t = 4\left(\frac{3}{2}\right)^t - 2$

16. (a) $x = Ae^{-3t}$ (b) $x = Ae^{-4t}+3$ (c) $x = 1/(Ae^{-3t}-4)$ and $x \equiv 0$. (d) $x = Ae^{-\frac{1}{3}t}$ (e) $x = Ae^{-2t}+5/3$

(f) $x = 1/(Ae^{-\frac{1}{2}t}-2)$ and $x \equiv 0$.

17. (a) $x = 1/(C-\frac{1}{2}t^2)$ and $x(t) \equiv 0$. (b) $x = Ce^{-3t/2}-5$ (c) $x = Ce^{3t}-10$ (d) $x = Ce^{-5t}+2t-\frac{2}{5}$

(e) $x = Ce^{-t/2}+\frac{2}{3}e^t$ (f) $x = Ce^{-3t}+\frac{1}{3}t^2-\frac{2}{9}t+\frac{2}{27}$

18. (a) $V(x) = (V_0+b/a)e^{-ax}-b/a$ (b) $V(x^*) = 0$ yields $x^* = (1/a)\ln(1+aV_0/b)$.

(c) $0 = V(\hat{x}) = (V_m+b/a)e^{-a\hat{x}}-b/a$ yields $V_m = (b/a)(e^{a\hat{x}}-1)$.

(d) $x^* = (1/0.001)\ln(1+0.001\cdot 12\,000/8) \approx 916$, and $V_m = (8/0.001)(e^{0.001\cdot 1200}-1) = 8000(e^{1.2}-1) \approx 18\,561$.

Chapter 12

12.1

1. (a) 2×2 (b) 2×3 (c) $m\times n$

2. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $u = 3$ and $v = -2$. (Equating the elements in row 1 and column 3 gives $u = 3$. Then, equating those in row 2 and column 3 gives $u - v = 5$ and so $v = -2$. The other elements then need to be checked, but this is obvious.)

12.2

1. Equations (a), (c), (d), and (f) are linear in x, y, z, and w, whereas (b) and (e) are nonlinear in these variables.

2. Yes: with x_1, y_1, x_2, and y_2 all constants, the system is linear in a, b, c, and d.

3. The three rows are $2x_1 + 4x_2 + 6x_3 + 8x_4 = 2$, $5x_1 + 7x_2 + 9x_3 + 11x_4 = 4$, and $4x_1 + 6x_2 + 8x_3 + 10x_4 = 8$.

4. The system is $\begin{cases} x_2 + x_3 + x_4 = b_1 \\ x_1 + x_3 + x_4 = b_2 \\ x_1 + x_2 + x_4 = b_3 \\ x_1 + x_2 + x_3 = b_4 \end{cases}$ with solution $\begin{cases} x_1 = -\frac{2}{3}b_1 + \frac{1}{3}(b_2 + b_3 + b_4) \\ x_2 = -\frac{2}{3}b_2 + \frac{1}{3}(b_1 + b_3 + b_4) \\ x_3 = -\frac{2}{3}b_3 + \frac{1}{3}(b_1 + b_2 + b_4) \\ x_4 = -\frac{2}{3}b_4 + \frac{1}{3}(b_1 + b_2 + b_3) \end{cases}$

(Adding the 4 equations, then dividing by 3, gives $x_1 + x_2 + x_3 + x_4 = \frac{1}{3}(b_1 + b_2 + b_3 + b_4)$. Subtracting each of the original equations in turn from this new equation gives the solution for $x_1, \ldots, x_4$. An alternative solution method is to eliminate the variables systematically, starting with (say) x_4.)

5. (a) The commodity bundle owned by individual j. (b) $a_{i1} + a_{i2} + \cdots + a_{in}$ is the total amount of commodity i owned by all individuals. The first case is when $i = 1$. (c) $p_1 a_{1j} + p_2 a_{2j} + \cdots + p_m a_{mj}$

6. After dropping terms with zero coefficients, the equations are $\begin{cases} -0.712Y + C = 95.05 \\ X - Y - S + C = 0.00 \\ 0.158X - S + 0.158C = 34.30 \\ X = 93.53 \end{cases}$

The solution is $X = 93.53$, $Y \approx 482.11$, $S \approx 49.73$, and $C \approx 438.31$.

12.3

1. $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 7 & 5 \end{pmatrix}$, $3\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 6 & 9 \end{pmatrix}$

2. $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 4 & 16 \end{pmatrix}$, $\mathbf{A} - \mathbf{B} = \begin{pmatrix} -1 & 2 & -6 \\ 2 & 2 & -2 \end{pmatrix}$, and $5\mathbf{A} - 3\mathbf{B} = \begin{pmatrix} -3 & 8 & -20 \\ 10 & 12 & 8 \end{pmatrix}$

12.4

1. $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\mathbf{a} - \mathbf{b} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$, $2\mathbf{a} + 3\mathbf{b} = \begin{pmatrix} 13 \\ 10 \end{pmatrix}$, and $-5\mathbf{a} + 2\mathbf{b} = \begin{pmatrix} -4 \\ 13 \end{pmatrix}$

2. $\mathbf{a} + \mathbf{b} + \mathbf{c} = (-1, 6, -4)$, $\mathbf{a} - 2\mathbf{b} + 2\mathbf{c} = (-3, 10, 2)$, $3\mathbf{a} + 2\mathbf{b} - 3\mathbf{c} = (9, -6, 9)$

3. By definition of vector addition and scalar multiplication, the left-hand side of the equation is the vector $3(x, y, z) + 5(-1, 2, 3) = (3x - 5, 3y + 10, 3z + 15)$. For this to equal the vector $(4, 1, 3)$, all three components must be equal. So the vector equation is equivalent to the equation system $3x - 5 = 4$, $3y + 10 = 1$, and $3z + 15 = 3$, with the obvious solution $x = 3$, $y = -3$, $z = -4$.

4. Here $\mathbf{x} = \mathbf{0}$, so for all i, the ith component satisfies $x_i = 0$.

5. Nothing, because $0 \cdot \mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$.

6. We need to find numbers t and s such that $t(2, -1) + s(1, 4) = (4, -11)$. This vector equation is equivalent to $(2t + s, -t + 4s) = (4, -11)$. Equating the two components gives the system (i) $2t + s = 4$; (ii) $-t + 4s = -11$. This system has the solution $t = 3$, $s = -2$, so $(4, -11) = 3(2, -1) - 2(1, 4)$.

$-2\mathbf{x} = 7\mathbf{a} + 8\mathbf{b} - \mathbf{a}$, so $2\mathbf{x} = 6\mathbf{a} + 8\mathbf{b}$, and $\mathbf{x} = 3\mathbf{a} + 4\mathbf{b}$.

$\mathbf{a} = 5$, $\mathbf{a}\cdot\mathbf{b} = 2$, and $\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}) = 7$. We see that $\mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot(\mathbf{a}+\mathbf{b})$.

he inner product of the two vectors is $x^2 + (x-1)x + 3\cdot 3x = x^2 + x^2 - x + 9x = 2x^2 + 8x = 2x(x+4)$, which s 0 for $x = 0$ and for $x = -4$.

(a) $\mathbf{x} = (5, 7, 12)$ (b) $\mathbf{u} = (20, 18, 25)$ (c) $\mathbf{u}\cdot\mathbf{x} = 526$

(a) The firm's revenue is $\mathbf{p}\cdot\mathbf{z}$. Its costs are $\mathbf{p}\cdot\mathbf{x}$. (b) Profit = revenue − costs. This equals $\mathbf{p}\cdot\mathbf{z} - \mathbf{p}\cdot\mathbf{x} = \mathbf{p}\cdot(\mathbf{z}-\mathbf{x}) = \mathbf{p}\cdot\mathbf{y}$. If $\mathbf{p}\cdot\mathbf{y} < 0$, then the firm makes a loss equal to $-\mathbf{p}\cdot\mathbf{y}$.

2. (a) Input vector $= \begin{pmatrix}0\\1\end{pmatrix}$ (b) Output vector $= \begin{pmatrix}2\\0\end{pmatrix}$ (c) Cost $= (1,3)\begin{pmatrix}0\\1\end{pmatrix} = 3$ (d) Revenue $= (1,3)\begin{pmatrix}2\\0\end{pmatrix} = 2$

(e) Value of net output $= (1,3)\begin{pmatrix}2\\-1\end{pmatrix} = 2 - 3 = -1$. (f) Loss = cost − revenue = 3 − 2 = 1, so profit = −1.

12.5

1. (a) $\mathbf{AB} = \begin{pmatrix}0 & -2\\3 & 1\end{pmatrix}\begin{pmatrix}-1 & 4\\1 & 5\end{pmatrix} = \begin{pmatrix}0\cdot(-1)+(-2)\cdot 1 & 0\cdot 4+(-2)\cdot 5\\3\cdot(-1)+1\cdot 1 & 3\cdot 4+1\cdot 5\end{pmatrix} = \begin{pmatrix}-2 & -10\\-2 & 17\end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix}12 & 6\\15 & 3\end{pmatrix}$.

(b) $\mathbf{AB} = \begin{pmatrix}26 & 3\\6 & -22\end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix}14 & 6 & -12\\35 & 12 & 4\\3 & 3 & -22\end{pmatrix}$ (c) $\mathbf{AB}$ is not defined, whereas $\mathbf{BA} = \begin{pmatrix}-1 & 4\\3 & 4\\4 & 8\end{pmatrix}$

(d) $\mathbf{AB} = \begin{pmatrix}0 & 0 & 0\\0 & 4 & -6\\0 & -8 & 12\end{pmatrix}$ and $\mathbf{BA} = (16)$, a 1×1 matrix.

2. (i) $3\mathbf{A} + 2\mathbf{B} - 2\mathbf{C} + \mathbf{D} = \begin{pmatrix}-1 & 15\\-6 & -13\end{pmatrix}$ (ii) $\mathbf{AB} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ (iii) From (ii) it follows that $\mathbf{C}(\mathbf{AB}) = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$.

3. $\mathbf{A}+\mathbf{B} = \begin{pmatrix}4 & 1 & -1\\9 & 2 & 7\\3 & -1 & 4\end{pmatrix}$, $\mathbf{A}-\mathbf{B} = \begin{pmatrix}-2 & 3 & -5\\1 & -2 & -3\\-1 & -1 & -2\end{pmatrix}$, $\mathbf{AB} = \begin{pmatrix}5 & 3 & 3\\19 & -5 & 16\\1 & -3 & 0\end{pmatrix}$,

$\mathbf{BA} = \begin{pmatrix}0 & 4 & -9\\19 & 3 & -3\\5 & 1 & -3\end{pmatrix}$, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \begin{pmatrix}23 & 8 & 25\\92 & -28 & 76\\4 & -8 & -4\end{pmatrix}$

4. (a) $\begin{pmatrix}1 & 1\\3 & 5\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix}$ (b) $\begin{pmatrix}1 & 2 & 1\\1 & -1 & 1\\2 & 3 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}4\\5\\1\end{pmatrix}$ (c) $\begin{pmatrix}2 & -3 & 1\\1 & 1 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

5. (a) $\mathbf{A} - 2\mathbf{I} = \begin{pmatrix}0 & 2\\1 & 3\end{pmatrix}$. The matrix $\mathbf{C}$ must be 2×2.

With $\mathbf{C} = \begin{pmatrix}c_{11} & c_{12}\\c_{21} & c_{22}\end{pmatrix}$, we need $\begin{pmatrix}0 & 2\\1 & 3\end{pmatrix}\begin{pmatrix}c_{11} & c_{12}\\c_{21} & c_{22}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, or $\begin{pmatrix}2c_{21} & 2c_{22}\\c_{11}+3c_{21} & c_{12}+3c_{22}\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

The last matrix equation has the unique solution $c_{11} = -3/2$, $c_{12} = 1$, $c_{21} = 1/2$, and $c_{22} = 0$.

(b) $\mathbf{B} - 2\mathbf{I} = \begin{pmatrix}0 & 0\\3 & 0\end{pmatrix}$, so the first row of any product matrix $(\mathbf{B}-2\mathbf{I})\mathbf{D}$ must be $(0, 0)$.

So no matrix $\mathbf{D}$ can possibly satisfy $(\mathbf{B}-2\mathbf{I})\mathbf{D} = \mathbf{I}$.

6. The product **AB** is defined only if **B** has n rows. And **BA** is defined only if **B** has m columns. So **B** must be an $n \times m$ matrix.

7. $\mathbf{B} = \begin{pmatrix} w-y & y \\ y & w \end{pmatrix}$, for arbitrary y, w.

8. $\mathbf{T}(\mathbf{Ts}) = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.35 \\ 0.40 \end{pmatrix} = \begin{pmatrix} 0.2875 \\ 0.2250 \\ 0.4875 \end{pmatrix}$

12.6

1. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC} = \begin{pmatrix} 3 & 2 & 6 & 2 \\ 7 & 4 & 14 & 6 \end{pmatrix}$

2. The 1×1 matrix $(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz)$

3. It is straightforward to show that $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ are both equal to the 2×2 matrix $\mathbf{D} = (d_{ij})_{2\times 2}$, whose four elements are $d_{ij} = a_{i1}b_{11}c_{1j} + a_{i1}b_{12}c_{2j} + a_{i2}b_{21}c_{1j} + a_{i2}b_{22}c_{2j}$ for $i = 1, 2$ and $j = 1, 2$.

4. (a) $\begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & 9 \\ 1 & 3 & 3 \end{pmatrix}$ (b) $(1, 2, -3)$

5. (a) (i) Note that $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2 \neq \mathbf{A}^2 - \mathbf{B}^2$ unless $\mathbf{AB} = \mathbf{BA}$.

(ii) Similarly $(\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{AB} - \mathbf{BA} + \mathbf{B}^2 \neq \mathbf{A}^2 - 2\mathbf{AB} + \mathbf{B}^2$ unless $\mathbf{AB} = \mathbf{BA}$.

(b) Equality occurs in both (i) and (ii) if and only if $\mathbf{AB} = \mathbf{BA}$.

6. (a) Verify directly by matrix multiplication. (b) $\mathbf{AA} = (\mathbf{AB})\mathbf{A} = \mathbf{A}(\mathbf{BA}) = \mathbf{AB} = \mathbf{A}$, so **A** is idempotent. Then just interchange **A** and **B** to show that **B** is idempotent.

(c) As the induction hypothesis, suppose that $\mathbf{A}^k = \mathbf{A}$, which is true for $k = 1$. Then $\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \mathbf{AA} = \mathbf{A}$, which completes the proof by induction.

7. If $\mathbf{P}^3\mathbf{Q} = \mathbf{PQ}$, then $\mathbf{P}^5\mathbf{Q} = \mathbf{P}^2(\mathbf{P}^3\mathbf{Q}) = \mathbf{P}^2(\mathbf{PQ}) = \mathbf{P}^3\mathbf{Q} = \mathbf{PQ}$.

8. (a) Verify directly by matrix multiplication. (b) Given $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, it is enough to have $a + d = ad - bc = 0$ with a, b, c, d not all 0. One example is $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. (c) See SM.

12.7

1. $\mathbf{A}' = \begin{pmatrix} 3 & -1 \\ 5 & 2 \\ 8 & 6 \\ 3 & 2 \end{pmatrix}$, $\mathbf{B}' = (0, 1, -1, 2)$, $\mathbf{C}' = \begin{pmatrix} 1 \\ 5 \\ 0 \\ -1 \end{pmatrix}$

2. $\mathbf{A}' = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}$, $\mathbf{B}' = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$, $(\mathbf{A}+\mathbf{B})' = \begin{pmatrix} 3 & 1 \\ 4 & 7 \end{pmatrix}$, $(\alpha\mathbf{A})' = \begin{pmatrix} -6 & 2 \\ -4 & -10 \end{pmatrix}$, $\mathbf{AB} = \begin{pmatrix} 4 & 10 \\ 10 & 8 \end{pmatrix}$,

$(\mathbf{AB})' = \begin{pmatrix} 4 & 10 \\ 10 & 8 \end{pmatrix} = \mathbf{B}'\mathbf{A}'$, and $\mathbf{A}'\mathbf{B}' = \begin{pmatrix} -2 & 4 \\ 10 & 14 \end{pmatrix}$.

Verifying the rules for transposition specified in Eqs (12.7.2)–(12.7.5) is now very easy.

3. Direct verification shows that for each of the two matrices the element in position ij equals the element in position ji for $i = 1, 2, 3$ and $j = 1, 2, 3$.

4. *Symmetry requires* $a^2 - 1 = a + 1$ and $a^2 + 4 = 4a$. The second equation has the unique root $a = 2$, which also satisfies the first equation.

5. *No! For example:* $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

6. $(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3)' = (\mathbf{A}_1(\mathbf{A}_2\mathbf{A}_3))' = (\mathbf{A}_2\mathbf{A}_3)'\mathbf{A}_1' = (\mathbf{A}_3'\mathbf{A}_2')\mathbf{A}_1' = \mathbf{A}_3'\mathbf{A}_2'\mathbf{A}_1'$. To prove the general case, use induction.

7. (a) Verify by direct multiplication. (b) $\begin{pmatrix} p & q \\ -q & p \end{pmatrix}\begin{pmatrix} p & -q \\ q & p \end{pmatrix} = \begin{pmatrix} p^2+q^2 & 0 \\ 0 & p^2+q^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow p^2 + q^2 = 1$.

(c) If $\mathbf{P}'\mathbf{P} = \mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$, then $(\mathbf{PQ})'(\mathbf{PQ}) = (\mathbf{Q}'\mathbf{P}')(\mathbf{PQ}) = \mathbf{Q}'(\mathbf{P}'\mathbf{P})\mathbf{Q} = \mathbf{Q}'\mathbf{I}_n\mathbf{Q} = \mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$.

8. (a) $\mathbf{TS} = \begin{pmatrix} p^3+p^2q & 2p^2q+2pq^2 & pq^2+q^3 \\ \frac{1}{2}p^3+\frac{1}{2}p^2+\frac{1}{2}p^2q & p^2q+pq+pq^2 & \frac{1}{2}pq^2+\frac{1}{2}q^2+\frac{1}{2}q^3 \\ p^3+p^2q & 2p^2q+2pq^2 & pq^2+q^3 \end{pmatrix} = \mathbf{S}$ because $p + q = 1$. A similar argument shows that $\mathbf{T}^2 = \frac{1}{2}\mathbf{T} + \frac{1}{2}\mathbf{S}$. To derive the formula for $\mathbf{T}^3$, multiply each side of the last equation on the left by $\mathbf{T}$.

(b) The appropriate formula is $\mathbf{T}^n = 2^{1-n}\mathbf{T} + (1 - 2^{1-n})\mathbf{S}$.

12.8

1. (a) The solution $x_1 = 5$, $x_2 = -2$ can be found by using Gaussian elimination to obtain

$$\begin{pmatrix} 1 & 1 & 3 \\ 3 & 5 & 5 \end{pmatrix} \begin{matrix} -3 \\ \end{matrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & -4 \end{pmatrix} \begin{matrix} \\ 1/2 \end{matrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \end{pmatrix} \begin{matrix} \\ -1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \end{pmatrix}$$

(b) Gaussian elimination yields

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 1 & -1 & 1 & 5 \\ 2 & 3 & -1 & 1 \end{pmatrix} \begin{matrix} -1 \quad -2 \\ \\ \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & 1 \\ 0 & -1 & -3 & -7 \end{pmatrix} \begin{matrix} \\ -1/3 \\ \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -1/3 \\ 0 & -1 & -3 & -7 \end{pmatrix} \begin{matrix} \\ 1 \quad -2 \\ \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 14/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & -3 & -22/3 \end{pmatrix} \begin{matrix} \\ \\ -1/3 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 1 & 14/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 22/9 \end{pmatrix} \begin{matrix} \\ \\ -1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & 20/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 22/9 \end{pmatrix}$$

The solution is therefore: $x_1 = 20/9$, $x_2 = -1/3$, $x_3 = 22/9$.

(c) The general solution is $x_1 = (2/5)s$, $x_2 = (3/5)s$, $x_3 = s$, where s is an arbitrary real number.

Using Gaussian elimination to eliminate x from the second and third equations, and then y from the third equation, we arrive at the augmented matrix $\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & a+5/2 & b-1/2 \end{pmatrix}$.

For any z, the first two equations imply that $y = -\frac{1}{2} + \frac{3}{2}z$ and $x = 1 - y + z = \frac{3}{2} - \frac{1}{2}z$.

rom the last equation we see that for $a \neq -\frac{5}{2}$, there is a unique solution with $z = (b - \frac{1}{2})/(a + \frac{5}{2})$.

or $a = -\frac{5}{2}$, there are no solutions if $b \neq \frac{1}{2}$, but there is one degree of freedom if $b = \frac{1}{2}$ (with z arbitrary).

r $c = 1$ and for $c = -2/5$ the solution is $x = 2c^2 - 1 + t$, $y = s$, $z = t$, $w = 1 - c^2 - 2s - 2t$, for arbitrary s and t.

other values of c there are no solutions.

e the first row down to row number three and use Gaussian elimination. There is a unique solution if and only if
3/4.

$\neq \frac{1}{4}b_3$, there is no solution. If $b_1 = \frac{1}{4}b_3$, there is an infinite set of solutions that take the form $x = -2b_2 + b_3 - 5t$,
$\frac{3}{2}b_2 - \frac{1}{2}b_3 + 2t$, $z = t$, with $t \in \mathbb{R}$.

12.9

1. $\mathbf{a}+\mathbf{b}=(3,3)$ and $-\frac{1}{2}\mathbf{a}=(-2.5,0.5)$. See Fig. A12.9.1.

2. (a) (i) $\lambda=0$ gives $\mathbf{x}=(-1,2)=\mathbf{b}$; (ii) $\lambda=1/4$ gives $\mathbf{x}=(0,7/4)$; (iii) $\lambda=1/2$ gives $\mathbf{x}=(1,3/2)$; (iv) $\lambda=3/4$ gives $\mathbf{x}=(2,5/4)$; (v) $\lambda=1$ gives $\mathbf{x}=(3,1)=\mathbf{a}$. See Fig. A12.9.2.

(b) As λ runs through $[0,1]$, the vector $\mathbf{x}$ traces out the line segment joining $\mathbf{b}$ to $\mathbf{a}$ in Fig. A12.9.2. (c) See SM.

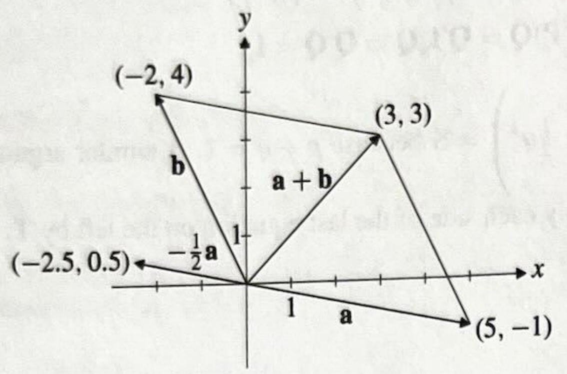

Figure A12.9.1

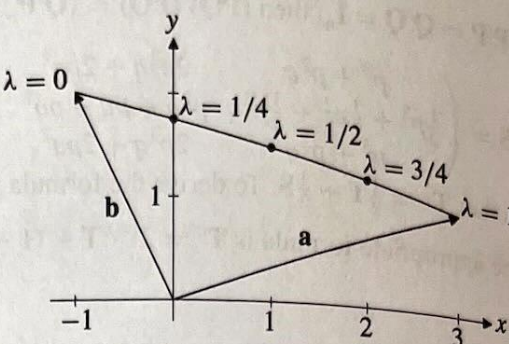

Figure A12.9.2

3. See Fig. A12.9.3.

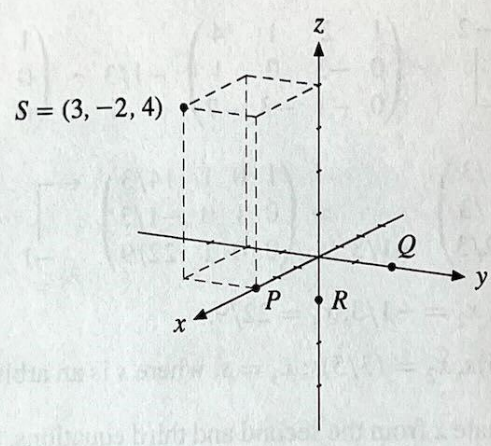

Figure A12.9.3

4. (a) A straight line through $(0,2,3)$ parallel to the x-axis.

(b) A plane parallel to the z-axis whose intersection with the xy-plane is the line $y=x$.

5. $\|\mathbf{a}\|=3$, $\|\mathbf{b}\|=3$, $\|\mathbf{c}\|=\sqrt{29}$. Also, $|\mathbf{a}\cdot\mathbf{b}|=6\le\|\mathbf{a}\|\cdot\|\mathbf{b}\|=9$.

6. (a) $x_1(1,2,1)+x_2(-3,0,-2)=(x_1-3x_2,2x_1,x_1-2x_2)=(5,4,4)$ when $x_1=2$ and $x_2=-1$.

(b) x_1 and x_2 would have to satisfy $x_1(1,2,1)+x_2(-3,0,-2)=(-3,6,1)$. Then $x_1-3x_2=-3$, $2x_1=6$, and $x_1-2x_2=1$. The first two equations imply that $x_1=3$ and $x_2=2$, which violate the last equation.

7. The pairs of vectors in (a) and (c) are orthogonal; the pair in (b) is not.

8. The vectors are orthogonal if and only if their inner product is 0. This is true if and only if $x^2-x-8-2x+x=x^2-2x-8=0$, which is the case for $x=-2$ and $x=4$.

9. If P is orthogonal and $\mathbf{c}_i$ and $\mathbf{c}_j$ are two different columns of $\mathbf{P}$, then $\mathbf{c}_i'\mathbf{c}_j$ is the element in row i and column j of $P'P = I$, so $\mathbf{c}_i'\mathbf{c}_j = 0$. If $\mathbf{r}_i$ and $\mathbf{r}_j$ are two different rows of $\mathbf{P}$, then $\mathbf{r}_i\mathbf{r}_j'$ is the element in row i and column j of $\mathbf{PP}' = \mathbf{I}$, so again $\mathbf{r}_i\mathbf{r}_j' = 0$.

10. $(\|\mathbf{a}\| + \|\mathbf{b}\|)^2 = \|\mathbf{a}\|^2 + 2\|\mathbf{a}\| \cdot \|\mathbf{b}\| + \|\mathbf{b}\|^2$, whereas $\|\mathbf{a}+\mathbf{b}\|^2 = (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+\mathbf{b}) = \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2$. Then $(\|\mathbf{a}\| + \|\mathbf{b}\|)^2 - \|\mathbf{a}+\mathbf{b}\|^2 = 2(\|\mathbf{a}\| \cdot \|\mathbf{b}\| - \mathbf{a} \cdot \mathbf{b}) \geq 0$ by the Cauchy–Schwarz inequality (12.9.7).

12.10

1. (a) $x_1 = 3t + 10(1-t) = 10 - 7t$, $x_2 = (-2)t + 2(1-t) = 2 - 4t$, and $x_3 = 2t + (1-t) = 1 + t$
(b) $x_1 = 1$, $x_2 = 3 - t$, and $x_3 = 2 + t$

2. (a) To show that $\mathbf{a}$ lies on L, put $t = 0$. (b) The direction of L is given by $(-1, 2, 1)$, and the equation of $\mathcal{P}$ is $(-1)(x_1 - 2) + 2(x_2 - (-1)) + 1 \cdot (x_3 - 3) = 0$, or $-x_1 + 2x_2 + x_3 = -1$.
(c) We must have $3(-t+2) + 5(2t-1) - (t+3) = 6$, and so $t = 4/3$. Thus $P = (2/3, 5/3, 13/3)$.

3. $x_1 - 3x_2 - 2x_3 = -3$

4. $2x + 3y + 5z \leq m$, with $m \geq 75$.

5. (a) This can be verified directly. (b) $(x_1, x_2, x_3) = (-2, 1, -1) + t(-1, 2, 3) = (-2 - t, 1 + 2t, -1 + 3t)$

Review exercises for Chapter 12

1. (a) $\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

2. (a) $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$ (b) $\mathbf{A} + \mathbf{B} - 2\mathbf{C} = \begin{pmatrix} -3 & -4 \\ -2 & -8 \end{pmatrix}$ (c) $\mathbf{AB} = \begin{pmatrix} -2 & 4 \\ 2 & -3 \end{pmatrix}$ (d) $\mathbf{C}(\mathbf{AB}) = \begin{pmatrix} 2 & -1 \\ 6 & -8 \end{pmatrix}$

(e) $\mathbf{AD} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$ (f) $\mathbf{DC}$ is not defined. (g) $2\mathbf{A} - 3\mathbf{B} = \begin{pmatrix} 7 & -6 \\ -5 & 5 \end{pmatrix}$ (h) $(\mathbf{A} - \mathbf{B})' = \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}$

(i) and (j): $(\mathbf{C}'\mathbf{A}')\mathbf{B}' = \mathbf{C}'(\mathbf{A}'\mathbf{B}') = \begin{pmatrix} -6 & 5 \\ -4 & 5 \end{pmatrix}$ (k) $\mathbf{D}'\mathbf{D}'$ is not defined. (l) $\mathbf{D}'\mathbf{D} = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 10 & 13 \\ 5 & 13 & 17 \end{pmatrix}$.

3. (a) $\begin{pmatrix} 2 & -5 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ 1 & 4 & 8 & 0 \\ 2 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ (c) $\begin{pmatrix} a-1 & 3 & -2 \\ a & 2 & -1 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$

$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 0 & -4 & 1 \\ 8 & 6 & 4 \\ -10 & 9 & 15 \end{pmatrix}$, $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 0 & 6 & -5 \\ -2 & 2 & 6 \\ -2 & 5 & 15 \end{pmatrix}$, $\mathbf{AB} = \begin{pmatrix} 13 & -2 & -1 \\ 0 & 3 & 5 \\ -25 & 74 & -25 \end{pmatrix}$,

$\mathbf{BA} = \begin{pmatrix} -33 & 1 & 20 \\ 12 & 6 & -15 \\ 6 & 4 & 18 \end{pmatrix}$, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 74 & -31 & -48 \\ 6 & 25 & 38 \\ -2 & -75 & -26 \end{pmatrix}$

The two matrix products on the left-hand side of the equation are $\begin{pmatrix} 2a+b & a+b \\ 2x & x \end{pmatrix}$ and $\begin{pmatrix} a & b \\ 2a+x & 2b \end{pmatrix}$. Equating heir difference $\begin{pmatrix} a+b & a \\ x-2a & x-2b \end{pmatrix}$ to the matrix $\begin{pmatrix} 2 & 1 \\ 4 & 4 \end{pmatrix}$ on the right-hand side yields the following four equalities: $+b = 2$, $a = 1$, $x - 2a = 4$, and $x - 2b = 4$. It follows that $a = b = 1$, $x = 6$.

6. (a) $\mathbf{A}^2 = \begin{pmatrix} a^2-b^2 & 2ab & b^2 \\ -2ab & a^2-2b^2 & 2ab \\ b^2 & -2ab & a^2-b^2 \end{pmatrix}$

(b) $(\mathbf{C}'\mathbf{B}\mathbf{C})' = \mathbf{C}'\mathbf{B}'(\mathbf{C}')' = \mathbf{C}'(-\mathbf{B})\mathbf{C} = -\mathbf{C}'\mathbf{B}\mathbf{C}$. So $\mathbf{A}$ is skew-symmetric if and only if $a = 0$.

(c) $\mathbf{A}_1' = \frac{1}{2}(\mathbf{A}' + \mathbf{A}'') = \frac{1}{2}(\mathbf{A}' + \mathbf{A}) = \mathbf{A}_1$, so $\mathbf{A}_1$ is symmetric. It is equally easy to prove that $\mathbf{A}_2$ is skew-symmetric, as well as that any square matrix $\mathbf{A}$ is therefore the sum $\mathbf{A}_1 + \mathbf{A}_2$ of a symmetric matrix $\mathbf{A}_1$ and a skew-symmetric matrix $\mathbf{A}_2$.

7. (a) $$\begin{pmatrix} 1 & 4 & 1 \\ 2 & 2 & 8 \end{pmatrix} \; -2 \sim \begin{pmatrix} 1 & 4 & 1 \\ 0 & -6 & 6 \end{pmatrix} -1/6 \sim \begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & -1 \end{pmatrix} -4 \sim \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \end{pmatrix}$$

The solution is $x_1 = 5$, $x_2 = -1$. (b) The solution is $x_1 = 3/7$, $x_2 = -5/7$, $x_3 = -18/7$.

(c) The solution is $x_1 = (1/14)x_3$, $x_2 = -(19/14)x_3$, where x_3 is arbitrary. (One degree of freedom.)

8. We use the method of Gaussian elimination:

$$\begin{pmatrix} 1 & a & 2 & 0 \\ -2 & -a & 1 & 4 \\ 2a & 3a^2 & 9 & 4 \end{pmatrix} \; 2 \;\; -2a \sim \begin{pmatrix} 1 & a & 2 & 0 \\ 0 & a & 5 & 4 \\ 0 & a^2 & 9-4a & 4 \end{pmatrix} -a \sim \begin{pmatrix} 1 & a & 2 & 0 \\ 0 & a & 5 & 4 \\ 0 & 0 & 9-9a & 4-4a \end{pmatrix}$$

For $a = 1$, the last equation is superfluous; the solution is $x = 3t - 4$, $y = -5t + 4$, $z = t$, with t arbitrary. If $a \neq 1$, we have $(9 - 9a)z = 4 - 4a$, so $z = 4/9$. The two other equations then become $x + ay = -8/9$ and $ay = 16/9$. If $a = 0$, there is no solution. If $a \neq 0$, the solution is $x = -8/3$, $y = 16/9a$, and $z = 4/9$.

9. Here $\|\mathbf{a}\| = \sqrt{35}$, $\|\mathbf{b}\| = \sqrt{11}$, and $\|\mathbf{c}\| = \sqrt{69}$. Moreover $|\mathbf{a}\cdot\mathbf{b}| = |(-1)\cdot 1 + 5 \cdot 1 + 3 \cdot (-3)| = |-5| = 5$. Then $\|\mathbf{a}\|\,\|\mathbf{b}\| = \sqrt{35}\sqrt{11} = \sqrt{385}$ is obviously greater than $|\mathbf{a}\cdot\mathbf{b}| = 5$, so the Cauchy–Schwarz inequality is satisfied.

10. Because $\mathbf{PQ} = \mathbf{QP} + \mathbf{P}$, multiplying on the left by $\mathbf{P}$ gives $\mathbf{P}^2\mathbf{Q} = (\mathbf{PQ})\mathbf{P} + \mathbf{P}^2 = (\mathbf{QP} + \mathbf{P})\mathbf{P} + \mathbf{P}^2 = \mathbf{QP}^2 + 2\mathbf{P}^2$. See SM for details of how to repeat this argument in order to prove by induction the result for higher powers of $\mathbf{P}$.

Chapter 13

13.1

1. (a) $3 \cdot 6 - 2 \cdot 0 = 18$ (b) $ab - ba = 0$ (c) $(2-x)(-x) - 1 \cdot 8 = x^2 - 2x + 8$ (d) $(a+b)^2 - (a-b)^2 = 4ab$ (e) $3^t 2^{t-1} - 3^{t-1}2^t = 3^{t-1}2^{t-1}(3-2) = 6^{t-1}$

2. See Fig. A13.1.2. The shaded parallelogram has area $3 \cdot 6 = 18 = \begin{vmatrix} 3 & 0 \\ 2 & 6 \end{vmatrix}$.

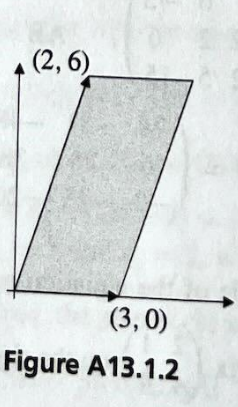

Figure A13.1.2